# Effects of Energy Addition and Dissipation on Dual-Spin Spacecraft Attitude Motion

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**The attitude motion of an axisymmetric dual-spin spacecraft in which energy is both added and dissipated is considered. Energy addition to the spacecraft occurs because a constant spin rate of the rotor with respect to the remainder of the spacecraft (i.e., the "platform") is maintained. Dissipation of energy is due to the operation of a spring-mass-dashpot nutation damper on the platform. The attitude motion is determined qualitatively and quantitatively. Qualitative results are obtained using the generalized method of averaging and energy-sink considerations. Quantitative information is obtained from the generalized method of averaging, numerical integration, and an extended energy-sink method. In applying the generalized method of averaging, it is assumed that the contribution of the damper mass to the overall transverse moment of inertia of the spacecraft is small. However, the nutation angle is not restricted. The results obtained using the generalized method of averaging agree well with numerical integration results. The extended energy-sink method produces an analytical result for the average time rate of change of the nutation angle that agrees with the result from the generalized method of averaging. This analysis substantiates the conclusion that energy-sink methods, when properly applied, lead to meaningful results even for systems containing driven rotors.**

## Introduction

DISAGREEMENT concerning the validity of the use of energy-sink methods of analysis for dual-spin spacecraft continues.[1-5] Furthermore, there appears to be some disagreement over exactly what an energy-sink method involves. It is certain, however, that it involves at least the assumption that the motion which causes dissipation is small enough that the spacecraft is at all times of interest a "quasirigid" body or a system of such bodies that has an essentially constant inertia ellipsoid. Also, in cases where an energy-sink method is applied to determine the effects of a particular dissipative device, the motion of the device is assumed to be affected by the attitude motion of the spacecraft with the device inoperable, and this zeroth-order motion of the device is used to determine energy dissipation rates. Finally, the energy dissipation rates are sometimes used to obtain analytical approximations to the spacecraft nutation angles.

Implicit in the use of such an approach is the assumption that the motion of the spacecraft is slowly affected by the operation of the device over a relatively long period of time. The resulting zeroth-order *uncoupling* of the attitude motion and device motion is extremely advantageous from an analysis viewpoint. It is also the type of behavior which appears in dynamic systems amenable to perturbation analyses, a fact pointed out and used by Cochran[6,7] and Alfriend and Hubert.[8]

Kane and Levinson[1] interpreted the energy-sink concept to imply that if an approximate expression for the nutation angle of a spacecraft containing a dissipative device can be found by considering only the spacecraft's rigid-body motion and if this expression contains the kinetic energy of the unperturbed rigid system, then the time history of the nutation angle can be predicted by inserting a realistic energy variation with time into the expression. They used Kane's[9] expression for the nutation angle of an axisymmetric gyrostat with a constant-speed (relative to the spacecraft proper) rotor and assumed time variations for the rotational kinetic energy to obtain time histories of the nutation angle. These were compared to time histories obtained by numerically integrating exact equations of motion of such a gyrostat, which also contained a spring-mass-dashpot nutation damper. Rather anomalistic results were obtained in that in some cases the exact nutation angle decreased while the rotational kinetic energy increased and vice versa. Furthermore, the authors of Ref. 1 found that if they used the exact rigid-body rotational kinetic energy as calculated from numerically exact angular velocity components in their approximate expressions for the nutation angle, then the prediction of nutation angle behavior was still not correct.

Kane and Levinson[1] concluded that energy-sink methods should not be used in dealing with systems containing driven rotors.

On the other hand, Hubert[3] has, primarily on the basis of numerical experiments, concluded that for a system containing a driven rotor and a dissipative device on the spacecraft proper (but none on the rotor) under certain defined conditions a decrease will occur in the rotational kinetic energy of the system, less that due to the rotor's rotation relative to the spacecraft proper, and that the nutation angle will also decrease. This energy, which Hubert calls the "core energy," is that of the spacecraft with the rotor fixed within the spacecraft. Hubert also presents results in Ref. 10 (p. 62) which indicate that the total energy of his system may increase as the nutation angle shows a net decrease. It should be noted that the operation of Hubert's dissipative device does not change his system's inertia properties.

The important point not addressed sufficiently in Refs. 1, 3, and 10, nor elsewhere apparently, is that if the nutation angle changes with time, energy must either be input to or removed from the system in order to maintain the rotor at a constant relative speed. This is usually done by using an electric motor.

The first and second specific purposes of this paper are to provide a rigorous deviation of a first-order solution for the nutational motion of the spacecraft model of Ref. 1, using the

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generalized method of averaging (GMA) and to show that this expression can be used to predict qualitatively the nutational motion of a system containing a driven rotor. The third specific purpose is to show that if account is taken of the change in the rotational kinetic energy due to rotor speed control, then use of an energy-sink method produces the same results for long-term nutational motion as the GMA. The overall purpose is to define more adequately the conditions under which energy-sink methods of analysis should produce meaningful results.

## Spacecraft Model

As mentioned above, the spacecraft model used in this paper is that of Ref. 1. Our representation of it is shown in Fig. 1. The model is composed of three bodies labeled D, P, and R, indicating the damper, platform, and rotor, respectively. Body D is a point mass that is constrained to move along a line fixed in the platform parallel to the $x_1$ axis. The mass of D is $m_D$ and its motion is resisted by a linear spring (constant $\sigma$) and dashpot (constant $\delta$). The distance from the $x_1$ axis to the body D is $b$. Body P is a rigid body of mass $m_P$, its mass distribution is such that when D is located on the $x_2$ axis, the inertia ellipsoid of the system, including the axisymmetric rotor R (mass $m_R$) is uniaxial. The rotor has its axis of rotation, which is also its axis of symmetry, parallel to the $x_1$ axis. The origin of the $Gx_1x_2x_3$ coordinate system coincides with the combined center of mass of bodies P and R. The principal centroidal axes of inertia of the spacecraft when D is on the $x_2$ axis are parallel to the axes $x_j$.

As in Ref. 1, we let $J_1$ and $J_2$ denote the centroidal principal moments of inertia of the spacecraft when there is no displacement of body D. Also, $\omega_j$ is the $x_j$ component of the angular velocity of body P, while the angular velocity of R with respect to P is denoted by $\mathbf{\Omega}$. The moment of inertia of body P about its symmetry axis is $J$.

### Exact Mathematical Model

The exact equations of motion for the spacecraft are given in Appendix A. They are nonlinear; hence, the analysis alternatives are to integrate them numerically, to perform a stability analysis, and to obtain approximate solutions to the equations. Also given in Appendix A are expressions for the kinetic and potential energies of the system.

### Approximate Mathematical Model

The angular momentum of the system is denoted $\boldsymbol{H}$. The same notation is used for the column matrix of the $x_j$ components of $\boldsymbol{H}$, viz. $H=(H_1H_2H_3)^T$. In terms of the $\omega_j$, $h \triangleq J\Omega$, $\dot{q}$, $q$, and system constants,

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \begin{bmatrix} J_1 & -\mu bq & 0 \\ -\mu bq & J_2+\mu q^2 & 0 \\ 0 & 0 & J_2+\mu q^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} h \\ 0 \\ -\mu b\dot{q} \end{bmatrix} \tag{1}$$

We note that $H_1 = H\cos\Theta$, $H_2 = H\sin\Theta\sin\Phi$, and $H_3 = H\sin\Theta\cos\Phi$, where $H = |\boldsymbol{H}|$, $\Theta$ is the nutation angle, and $\Phi$ is the angle of "proper motion" or spin of body $P$. Let $\eta = q/b$ and $\epsilon = \mu b^2/J_2$, where $\mu = m_D m_G/(m_D + m_G)$ and $m_G = m_D + m_R$. Then, through first order in $\epsilon$, we may write,

$$\omega_1 = (H_1 - h)/J_1 + \epsilon\eta H_2/J_1 \tag{2a}$$

$$\omega_2 = H_2/J_2 + \epsilon[\eta(H_1-h)/J_1 - \eta^2 H_2/J_2] \tag{2b}$$

$$\omega_3 = H_3/J_2 + \epsilon[\dot{\eta} - \eta^2 H_3/J_2] \tag{2c}$$

Because the effects of external torques are neglected herein, we have (exactly),

$$\dot{H} = \tilde{H}\omega \tag{3}$$

where

$$\tilde{H} = \begin{bmatrix} 0 & -H_3 & H_2 \\ H_3 & 0 & -H_1 \\ -H_2 & H_1 & 0 \end{bmatrix} \tag{4}$$

If we use the approximations for the $\omega_j$ given by Eqs. (2) in Eq. (3), we get

$$\dot{H}_1 \simeq -\epsilon[\eta(H_1-h)H_3/J_1 - \dot{\eta}H_2] \tag{5a}$$

$$\dot{H}_2 \simeq [H_1(J_2-J_1)/J_2 - h]H_3/J_1 + \epsilon[\eta H_2H_3/J_1 - \dot{\eta}H_1 + \eta^2 H_1H_3/J_2] \tag{5b}$$

$$\dot{H}_3 \simeq -[H_1(J_2-J_1)/J_2 - h]H_2/J_1 + \epsilon[\eta(H_1-h)H_1/J_1 - \eta^2 H_1H_2/J_2 - \eta H_2^2/J_2] \tag{5c}$$

Because of Eq. (3), even though $\omega = (\omega_1\omega_2\omega_3)^T$ was approximated, Eqs. (5) have the integral $H_1^2 + H_2^2 + H_3^2 = H^2$, where $H$ is constant.

An equation for $\eta$ is needed to complete this set of approximate equations. It turns out that $\eta$ is needed only to zeroth order [provided $\eta$ is $\mathcal{O}(1)$] since it appears only in the first-order parts of Eqs. (5). Referring to Appendix A [Eq. (A2)] and using Eqs. (2) and (5), we find

$$\ddot{\eta} + (\delta/\mu)\dot{\eta} + [(\sigma/\mu) - (H_2^2+H_3^2)/J_2^2]\eta = [H_1/J_2 - 2(H_1-h)/J_1]H_2/J_2 + \mathcal{O}(\epsilon) \tag{6}$$

Equations (5) and (6) constitute a first-order approximation to the equations of motion. By using the exact expression for the kinetic energy given by Eqs. (A3) and (2), the following expression for the zeroth-order part of the total energy $E$ of the system may be obtained:

$$E_0 = \tfrac{1}{2}[(H_1-h)^2/J_1 + (H_2^2+H_3^2)/J_2] + \tfrac{1}{2}[h^2/J + 2(H_1-h)h/J_1] \tag{7}$$

No terms containing $\eta$ or $\dot{\eta}$ appear explicitly in $E_0$ (see Appendix B).

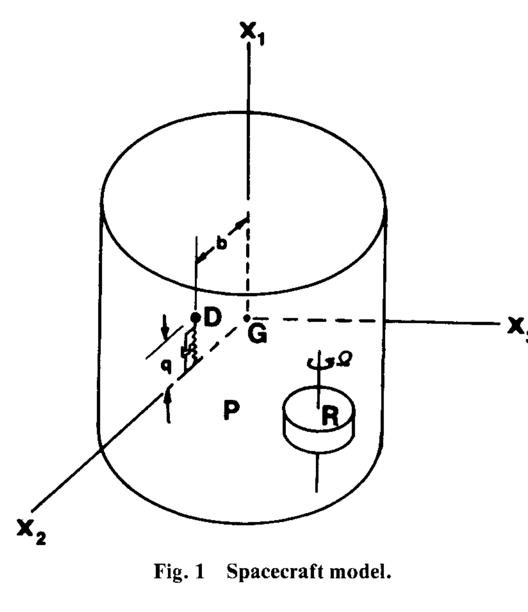


**Fig. 1 Spacecraft model.**

## Energy Changes due to Rotor Speed Control

Consider for the present that the damper mass is fixed, i.e., $\eta \equiv 0$. Then, the total energy of the system can be expressed as

$$E_0 = \tfrac{1}{2}[(H_1 - P_\alpha)^2/I_1 + (H_2^2 + H_3^2)/J_2 + P_\alpha^2/J] \tag{8}$$

where $P_\alpha = J(\omega_1 + \Omega)$ is the $x_1$ component of the total angular momentum of body R and $I_1$ is the moment of inertia of the bodies P and D about their principal axis which is parallel to $x_1$.

Now, in the energy-sink context, suppose that the energy is expended, or absorbed, to maintain $h$ constant and additional energy is dissipated in the system so that

$$\dot{E}_0 = T_\alpha \Omega + \dot{K}_D \tag{9}$$

where $T_\alpha \Omega$ is the power expended, or absorbed, to keep $h$ constant and $\dot{K}_D$ is the rate at which additional energy is dissipated, i.e., $\dot{K}_D < 0$.

For $\eta \equiv 0$ we have

$$P_\alpha = (I_1 h + J H_1)/J_1 \tag{10}$$

and, for later use,

$$(H_1 - P_\alpha) = I_1(H_1 - h)/J_1 \tag{11}$$

Thus,

$$\dot{P}_\alpha = (J/J_1)\dot{H}_1 \tag{12}$$

and, since $\dot{P}_\alpha = T_\alpha$ and $\Omega = h/J$,

$$\dot{E}_0 = (h/J_1)\dot{H}_1 + \dot{K}_D \tag{13}$$

It follows from Eq. (13) that if $h > 0$ and $\dot{H}_1 > 0$, which is the case if $\Theta$ is decreasing, then energy must be input to keep $h$ constant.

Differentiating $E_0$ explicitly with respect to time and using $H_2^2 + H_3^2 = H^2 - H_1^2$ and Eqs. (10-12) results in

$$\dot{H}_1(H_1 - h)/J_1 - \dot{H}_1 H_1/J_2 = \dot{K}_D \tag{14}$$

or

$$\dot{H}_1 = \dot{K}_D / \{[H_1(J_2 - J_1)/J_2 - h]/J_1\} \tag{15}$$

Thus, $\dot{H}_1 > 0$ if

$$H_1(J_2 - J_1)/J_2 - h < 0 \tag{16}$$

The inequality of Eq. (16) is Hubert's[3] criterion for asymptotic stability of the steady state, $H_1 = H$, $H_2 = H_3 = 0$, $\Theta = 0$, in the case of an axisymmetric "core body." In fact, the "core energy" for our system is simply

$$K_c = \tfrac{1}{2}[(H_1 - h)^2/J_1 + (H^2 - H_1^2)/J_2] \tag{17}$$

Clearly, $\dot{K}_c = \dot{K}_D$.

If we do not take into account the torque on the rotor, then instead of Eq. (15), we get

$$\dot{H}_1 = \dot{E}_0 / [H_1(J_2 - J_1)/(J_1 J_2)] \tag{18}$$

For $H_1 > 0$, $\dot{H}_1$ has the sign of $\dot{E}_0/(J_2 - J_1)$.

From the above, we conclude that, in the case of a driven rotor, the energy which decreases due to energy dissipation of an energy-sink type is the *core energy*. The energy $E_0$ may increase or decrease, depending on whether the energy input to the rotor is greater or less than that dissipated. We show *infra* that, on the average, Eq. (15) leads to the same result as the generalized method of averaging.

## Application of the Generalized Method of Averaging (GMA)

Equations (5) are not in the normal form for the application of the GMA. However, by using the equations $H_2 = (H^2 - H_1^2)^{1/2} \sin\Phi$ and $H_3 = (H^2 - H_1^2)^{1/2} \cos\Phi$, we may put Eq. (5a) into the form,

$$\dot{H}_1 = -\epsilon(H^2 - H_1^2)^{1/2}\{[\eta(H_1 - h)/J_2]\cos\Phi - \dot{\eta}\sin\Phi\} \tag{19}$$

and may show that

$$\dot{\Phi} = [(J_2 - J_1)H_1/J_2 - h]/J_1 + \mathcal{O}(\epsilon) \tag{20}$$

For convenience, we let $x = H_1$ and $y = \Phi$. Then, we can use the above equation for $H_2$ in Eq. (6) and construct a particular "steady-state" solution of $\eta$ of the form,

$$\eta = C_0(x)\cos y + D_0(x)\sin y + \mathcal{O}(\epsilon) \tag{21}$$

where

$$C_0(x) = -c g_0(x) A(x)/\Delta(x) \tag{22a}$$

$$D_0(x) = [\lambda_D^2(x) - g_0^2(x)]A(x)/\Delta(x) \tag{22b}$$

$$g_0(x) = [(J_2 - J_1)x/J_2 - h]/J_1 \tag{22c}$$

$$A(x) = [x/J_2 - 2(h - x)/J_1](H^2 - x^2)^{1/2}/J_2 \tag{22d}$$

$$\lambda_D^2 = \sigma/\mu - (H^2 - x^2)/J_2^2 \tag{22e}$$

$$\Delta(x) = [\lambda_D^2(x) - g_0^2(x)]^2 + c^2 g_0^2(x) \tag{22f}$$

$$c = \delta/\mu \tag{22g}$$

Then, after some algebra, we obtain a "steady-state" first-order equation for $x$, viz.,

$$\dot{x} = -\tfrac{1}{2}\epsilon\{c g_0(x) A^2(x) J_2/\Delta(x) + (H^2 - x^2)^{1/2}(x/J_2)[C_0(x)\cos 2y + D_0(x)\sin 2y]\} \tag{23}$$

The corresponding equation for $y$ has the form,

$$\dot{y} = g_0(x) + \epsilon[h(x) + a(x)\cos 2y + b(x)\sin 2y] \tag{24}$$

For a first-order solution for $x$, we do not need $h(x)$, $a(x)$, and $b(x)$ explicitly unless we are concerned with the phase change caused by first-order terms.

The first-order solution for $x$ (see Ref. 7) is

$$x = \bar{x} + \epsilon u_1(\bar{x}, \bar{y}) \tag{25}$$

where $\bar{x}$ is obtained from

$$\dot{\bar{x}} = -\tfrac{1}{2}\epsilon c g_0(\bar{x}) A^2(\bar{x}) J_2/\Delta(\bar{x}) \tag{26}$$

and

$$u_1(\bar{x}, \bar{y}) = -\tfrac{1}{2}(H^2 - \bar{x}^2)^{1/2}(\bar{x}/J_2)[C_0(\bar{x})\sin 2\bar{y} - D_0(\bar{x})\cos 2\bar{y}] \tag{27}$$

We note that $\dot{\bar{x}} > 0$ if $A(\bar{x}) \neq 0$ and $g_0(\bar{x}) < 0$. The latter may be interpreted as requiring retrograde "proper motion," or spin, for body P, the "core." The condition $A(\bar{x}) = 0$ is approximately satisfied for small $\Theta$ when minus twice the spin rate is equal to the rate of precession of body P.

The solution of Eq. (26) can be obtained by separating variables and using partial fractions. It has the form,

$$
\begin{aligned}
t-t_0 = A_6\{ & A_1 \ln[(a_1-\bar{x}_0)/(a_1-\bar{x})] \\
& + A_2 \ln[(a_1+\bar{x})/(a_1+\bar{x}_0)] \\
& + A_3[1/(\bar{x}_0-a_2)-1/(\bar{x}-a_2)] \\
& + A_4 \ln[(\bar{x}-a_2)/(\bar{x}_0-a_2)] \\
& + A_5 \ln[(a_3-\bar{x}_0)/(a_3-\bar{x})]\}
\end{aligned} \tag{28}
$$

where $a_1=H$, $a_2=[2J_2(2J_2-J_1)]h$, and $a_3=[J_2/(J_2-J_1)]h$ are the zeros of Eq. (26); $A_6=2J_2^4J_1^3/[c\epsilon(2J_2-J_1)^2(J_2-J_1)]$ and the $A_j$ $(j=1,2,...,5)$ are the partial fraction coefficients.

For small values of $\Theta$, and hence $\bar{\Theta}$, Eq. (26) can be replaced by

$$\dot{\bar{\Theta}} = \tfrac{1}{2}\{\epsilon c g_0(H)[H/J_2-2g_0(H)]^2 J_2/\Delta(H)\}\bar{\Theta} \tag{29}$$

If the system is considered rigid, then the rate at which its symmetry axis precesses about the angular momentum vector is $\dot{\psi}=H/J_2$. Hence, as mentioned above and in Ref. 5, nutation damping *theoretically* ceases when $\dot{\Psi}=2\dot{\Phi}$. *Practically,* only near commensurability of these rates should occur, but such an event will produce small values of the magnitude of $\dot{\Theta}$.

We also note that the results for the mass-spring damper in Ref. 8 and that given in Eq. (23) are the same when identical assumptions are made. However, in Refs. 5 and 8, the assumption of small nutation angle was made. Here $\Theta$ may be large.

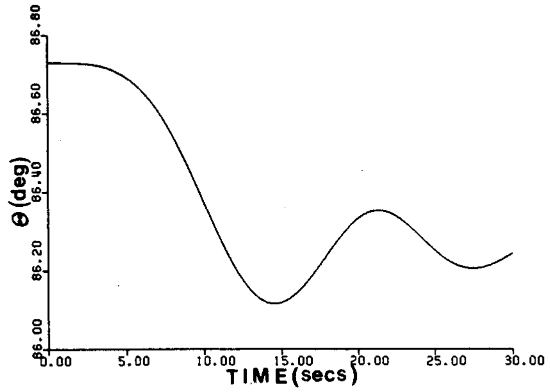


**Fig. 2 Nutation angle time history for case I.**

## Energy-Sink Solution for $H_1$

To see how the energy-sink approach compares with the foregoing, more rigorous analysis, we note that the total time rate of change of the energy of the system minus that due to maintaining $h$ constant is $-\delta b^2\dot{\eta}^2$. Neglecting the contribution of the displacement of the damper mass to total energy change, we put $\dot{K}_c=\dot{K}_D$ and get

$$\dot{K}_c = -\delta b^2\dot{\eta}^2$$

From Eq. (15) and the steady-state solution for $\eta$ we get for the averaged (over one period in $\Phi$) time rate of change of $x=H_1$

$$\dot{\bar{x}} = -c\mu b^2 g_0(\bar{x})[C_0^2(\bar{x})+D_0^2(\bar{x})]/2$$

or

$$\dot{\bar{x}} = -c\epsilon g_0(\bar{x})A^2(\bar{x})J_2/(2\Delta(\bar{x}))$$

which is identical to Eq. (26). Therefore, if the core energy is used in the energy-sink approach, the result for qualitative motion in $\Theta$ is the same as that found via the more rigorous GMA.

## Numerical Results

To determine how well our approximate analytical results predict the behavior of a spacecraft model, we take Kane and Levinson's[1] data (note that our notation is slightly different): $m_D=10$ kg, $m_G=990$ kg, $b=1$ m, $\delta=2$ N·s/m, $\sigma=10$ N/m, 2 $N$·s/m, $J_1=100$ kg·m², $J_2=175$ kg·m², $\Omega=10$ rad/s, and $\dot{q}=q=0$ at $t=0$.

The first example of Ref. 1 has $\omega_1=\omega_2=0$ and $\omega_3=1$ rad/s at $t=0$. We call this case I. The value of $g_0(x)$ at $t=0$ is approximately $-0.0571$ rad/s; hence, $\Theta$ should decrease. Numerical integration of the exact equations of the motion yields a nutation angle/time history which supports this conclusion (see Fig. 2). The short-period oscillations are due to transient motion of the damper mass. When numerical integration of the approximate equations (5) and (6) was attempted, the stiffness of the spring of the damper was found to be insufficient to restrain the damper mass. By using a stiffer spring ($\delta=20$ N/m), consistent results were obtained.

A second example in Ref. 1 (our case II), has $\omega_1=1$ rad/s, $\omega_2=0.1$ rad/s, and $\omega_3=0$. For these values, $g_0(x)$ is initially *positive.* Hence, $\Theta$ should increase. By referring to Fig. 3, we see this is what happens.

The analytical solution for $\Theta$ for the second case is also shown in Fig. 3 along with the "exact" numerical solution. The analytical solution was obtained by using values of $\omega_1$, $\omega_2$, and $\omega_3$ at time $t=12.4$ s.

The final example of Ref. 1 (our case III) has $\omega_1=0$, $\omega_2=0$, and $\omega_3=0.01$ rad/s. For these values, $g_0(x)$ is the same as for case I. The nutation period is approximately $\pi/|g_0(H)|=55$ s. Also, the motion of the damper mass is small. Hence, we

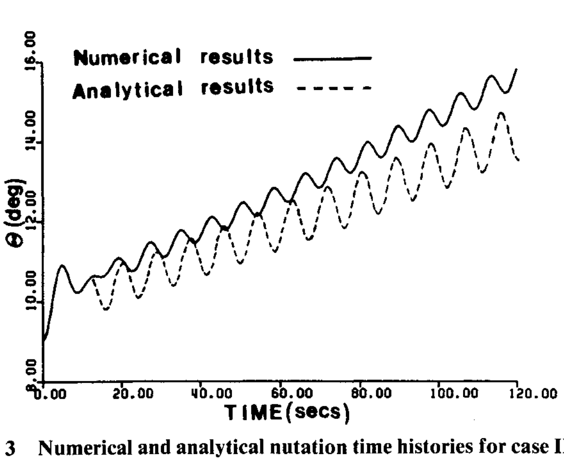


**Fig. 3 Numerical and analytical nutation time histories for case II.**

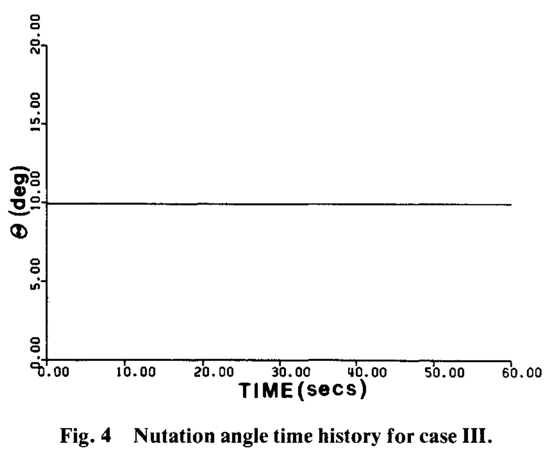


**Fig. 4 Nutation angle time history for case III.**

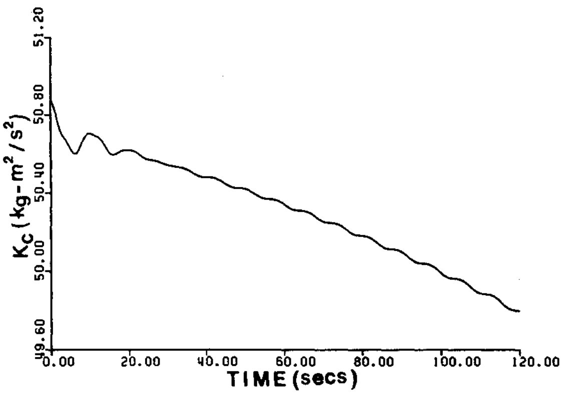


**Fig. 5 Core energy, case II.**

expect a decrease in $\Theta$, but only a very slow one. The time history of Fig. 4 shows no observable decrease. However, if a "soft" spring is used ($\sigma = 0.035$ N/m) so that the damper is tuned to the nutation frequency, then $\Theta$ decreases from about 9.92 to about 9.70 deg in 30 s.

In all of the cases the core energy ultimately decreased. Because of long nutation periods and very slow net changes in energy, the long-term behavior of $K_c$ may not be discovered if the duration of integration is short. In case II, for example, the ultimate nature of the change in $K_c$ is not certain until $t > 20$ s. (See Fig. 5.) This brings up the question of the length of time numerical integrations should be carried out. It is clear that the duration of integration for this particular system should be no less than several nutation periods when the damper mass motion is significant, if a true picture of the nutational motion of the spacecraft modeled is to be revealed.

## Conclusions

It has been shown that if the condition of small internal mass motion, which has always been implicit in the energy-sink concept, is satisfied and if due account is taken of the energy changes caused by maintaining the rotor of a dual-spin spacecraft at a constant speed, then the results obtained using an energy-sink method are qualitatively the same as those obtained using a more rigorous perturbation method. Here, by "qualitatively" we mean that the averaged time rates of change of the nutation angle are the same.

In obtaining the energy-sink result, we have shown, by eliminating the sign-variable power term due to rotor speed control from consideration, that Hubert's[3] hypothesis concerning the "core energy" is correct. That is, when energy dissipation is present on the "core" body, the core energy always decreases.

We wish to emphasize that neither perturbation nor energy-sink methods should be applied without first carefully considering whether or not the internal mass motion is sufficiently small to justify their use. Caution must also be exercised in reaching conclusions based on numerical solutions of limited extent.

## Appendix A: Exact Equations of Motion and Energy

The exact equations of motion are given by Kane and Levinson.[1] They are reproduced here in our notation. Also given is the exact total energy of the system.

### Equations of Motion

$$J_1\dot{\omega}_1 - \mu b q\dot{\omega}_2 + \mu b(\omega_3\omega_1 q - 2\omega_2\dot{q}) = 0 \tag{A1}$$

$$\mu b q\dot{\omega}_1 - (J_2+\mu q^2)\dot{\omega}_2 + \mu q(\omega_3\omega_1 q + \omega_2\omega_3 b - 2\omega_2\dot{q}) + (J_2 - J_1)\omega_3\omega_1 - J\Omega\omega_3 = 0 \tag{A2}$$

$$(J_2+\mu q^2)\dot{\omega}_3 - \mu b\ddot{q} + [(\omega_2^2-\omega_1^2)qb + \omega_1\omega_2 q^2 + 2q\dot{q}\omega_3] + (J_2-J_1)\omega_1\omega_2 - J\Omega\omega_2 = 0 \tag{A3}$$

$$\mu b\dot{\omega}_3 - \mu\ddot{q} + \mu[q(\omega_2^2+\omega_3^2) - \omega_1\omega_2 b] - \sigma q - \delta\dot{q} = 0 \tag{A4}$$

where $\mu = m_D m_G/(m_D + m_G)$.

### Total Energy

The kinetic energy is

$$K = \tfrac{1}{2}[J_1\omega_1^2 + J_2(\omega_2^2+\omega_3^2) + J\omega(\omega+2\omega_1)] + \tfrac{1}{2}\mu[\dot{q}^2 + (\omega_2^2+\omega_3^2)q^2 - 2\omega_1\omega_2 bq] \tag{A5}$$

The total potential energy is

$$V = \sigma q^2/2 \tag{A6}$$

See body of the paper for definitions of the symbols.

## Appendix B: Energy Consideration

The total energy $E$ of the system is given by Eq. (A4) plus $\tfrac{1}{2}\sigma q^2$. Let the nondimensional time $\tau = t(H/J_2)$ be introduced so that $\mathrm{d}(\ )/\mathrm{d}t = [\mathrm{d}(\ )/\mathrm{d}\tau](H/J_2)$ and let $(\ )' = \mathrm{d}(\ )/\mathrm{d}\tau$. Then $E$ can be written as

$$E = \tfrac{1}{2}[J_1\omega_1^2 + 2hJ_1\omega_1 + J_2(\omega_2^2+\omega_3^2)] + \tfrac{1}{2}\epsilon(H^2/J_2)\{(\eta')^2 - 2\eta'\omega_3 J_2/H + \eta^2[(\omega_2^2+\omega_3^2) + (\sigma/\mu)]J_2^2/H^2\} \tag{B1}$$

Next, use Eqs. (2) to approximate the $\omega_j$ to get

$$E = E_0 + \tfrac{1}{2}\epsilon(H^2/J_2)\{(\eta')^2 - 2\eta' H_3/H + \eta^2[(H^2-H_1^2) + (\sigma/\mu)]J_2^2/H^2\} + \mathcal{O}(\epsilon^2) \tag{B2}$$

Hence, $E$ may be written as

$$E = E_0 + \epsilon E_1 + \ldots \tag{B3}$$

Now, we have *exactly,*

$$\dot{E} = T_\alpha\Omega - \sigma\dot{q} \tag{B4}$$

or, since $\dot{E}_0 = T_\alpha\Omega + \dot{K}_c$,

$$\dot{K}_c + \epsilon\dot{E}_1 + \ldots = -\sigma\dot{q}^2 \tag{B5}$$

Assume now that quasisteady motion has been achieved in which the net change in any variable after a nutation cycle of period $T$ is at most $\mathcal{O}(\epsilon)$. Then,

$E_j(T) - E_j(0)$, $j > 0$, is $\mathcal{O}(\epsilon)$, and to $\mathcal{O}(\epsilon)$

$$K_c(T) - K_c(0) = -\int_0^T \sigma\dot{q}^2\,\mathrm{d}t \tag{B6}$$

Here, $T$ is to be computed using the value of $H_1$ at the beginning of the cycle at $t = 0$.

Note that the net change in *any* variable must be $\mathcal{O}(\epsilon)$ or less to guarantee that $E_1(T) - E_1(0)$ is negligible. If, for example, the change in $\eta$ is $\mathcal{O}(1/\epsilon^{1/2})$, then the terms in $E$ containing $\eta^2$ will render Eq. (B6) invalid. The damper spring must, of course, be stiff enough to keep this from happening.

Under the conditions required for the validity of Eq. (B6), the average value of the time rate of change of $H_1$, through $\mathcal{O}(\epsilon)$, can be found by fixing $H_1$ and performing the indicated integration.

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